

# THE PERRON-FROBENIUS THEOREM FOR MARKOV SEMIGROUPS

OMAR HIJAB

ABSTRACT. Let  $P_t^V$ ,  $t \geq 0$ , be the Schrodinger semigroup associated to a potential  $V$  and Markov semigroup  $P_t$ ,  $t \geq 0$ , on  $C(X)$ . Existence is established of a left eigenvector and right eigenvector corresponding to the spectral radius  $e^{\lambda_0 t}$  of  $P_t^V$ , simultaneously for all  $t \geq 0$ . This is derived with no compactness assumption on the semigroup operators.

## 1. INTRODUCTION

Let  $X$  be a compact metric space and let  $P_t$ ,  $t \geq 0$ , be a Markov semigroup on  $C(X)$  with generator  $L$ . Given  $V$  in  $C(X)$  let  $P_t^V$ ,  $t \geq 0$ , denote the Schrodinger semigroup on  $C(X)$  generated by  $L + V$ . Then the principal eigenvalue

$$\lambda_0(V) \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \log \|P_t^V\|$$

is given by the Donsker-Varadhan formula [4]

$$\lambda_0(V) = \sup_{\mu} \left( \int_X V d\mu - I(\mu) \right),$$

where the supremum is over probability measures  $\mu$  on  $X$ , and

$$I(\mu) \equiv - \inf_{u \in \mathcal{D}^+} \int_X \frac{Lu}{u} d\mu.$$

Here the infimum is over positive  $u$  in the domain  $\mathcal{D}$  of  $L$ .

An *equilibrium measure* is a measure  $\mu$  achieving the supremum in the Donsker-Varadhan formula. Let  $\lambda_0 = \lambda_0(V)$ .

A *ground state relative to  $\mu$*  is a Borel function  $\psi$  satisfying  $\psi > 0$  a.s.  $\mu$  and

$$e^{-\lambda_0 t} P_t^V \psi = \psi, \quad a.s. \mu, t \geq 0.$$

A *ground measure* is a measure  $\pi$  satisfying

$$(1.1) \quad \int_X e^{-\lambda_0 t} P_t^V f d\pi = \int_X f d\pi, \quad t \geq 0$$

for  $f$  in  $C(X)$ .

**Theorem 1.** *Suppose  $\pi$  and  $\mu$  are measures with  $\mu \ll \pi$ . Suppose also  $\psi = d\mu/d\pi$  satisfies  $\psi \log \psi \in L^1(\pi)$ . Then the following hold.*

- *If  $\pi$  is a ground measure and  $\psi$  is a ground state relative to  $\mu$ , then  $\mu$  is an equilibrium measure.*

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- If  $\pi$  is a ground measure and  $\mu$  is an equilibrium measure, then  $\psi$  is a ground state relative to  $\mu$ .
- If  $\mu$  is an equilibrium measure and  $\psi$  is a ground state relative to  $\mu$ , then  $\pi$  is a ground measure.

Here is the Perron-Frobenius Theorem in this setting.

**Theorem 2.** Fix  $V$  in  $C(X)$ , suppose

$$e^{-\lambda_0 t} \|P_t^V\| \leq C, \quad t \geq 0,$$

for some  $C > 0$ , and let  $\mu$  be an equilibrium measure. Then there is a ground measure  $\pi$  satisfying  $\mu \ll \pi$ ,  $\psi = d\mu/d\pi$  is a ground state relative to  $\mu$ , and  $\psi \log \psi \in L^1(\pi)$ .

This says there is a nonnegative right eigenvector  $\psi$  and a nonnegative left eigenvector  $\pi$ , corresponding to the spectral radius  $e^{\lambda_0 t}$  of  $P_t^V$ , simultaneously for all  $t \geq 0$ .

Neither the hypothesis nor the conclusion hold when  $L + V$  is a Jordan block: Take  $X = \{0, 1\}$  and

$$L = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In this case every measure  $\mu = (p, 1 - p)$  is an equilibrium measure and there is a unique ground measure  $\pi = (0, 1)$ .

In this generality, there is no guarantee of uniqueness of  $\mu$ ,  $\pi$  or  $\psi$ .

If  $P_t$ ,  $t \geq 0$ , is self-adjoint in  $L^2(\rho)$  for some measure  $\rho$  on  $X$ , then, under suitable conditions, the Donsker-Varadhan formula reduces [5] to the classical Rayleigh-Ritz formula for the principal eigenvalue, and every ground state  $\psi$  relative to  $\rho$  yields a ground measure  $d\pi = \psi d\rho$  and an equilibrium measure  $d\mu = \psi d\pi = \psi^2 d\rho$ . In the self-adjoint case, existence of ground states is classical (Gross [11]).

The Perron-Frobenius Theorem (with uniqueness) is known to hold in  $L^p$  for positive operators in these cases: for finite irreducible matrices (Friedland [7], Friedland-Karlin [8], Sternberg [19]), under a positivity improving property (Aida [1]), under a spectral gap condition (Gong-Wu [10]), and under uniform integrability or irreducibility (Wu [20]).

The existence of ground measures for positive operators and the existence of ground states for compact positive operators are classical results due to Krein-Rutman [13], see also Schaefer [16], [17]. In general, ground states do not exist pointwise everywhere on  $X$ , for example when  $L \equiv 0$ . The novelty of the above result is the handling of the general non-compact case by interpreting ground states as densities against ground measures, and the link with equilibrium measures.

The techniques used here are self-contained and follow the original papers [4], [5]. Preliminaries are discussed in section 2, equilibrium measures in section 3, ground measures and ground states in section 4, and entropy in section 5. The theorems are proved in section 6.

## 2. PRELIMINARIES

Let  $X$  be a compact metric space, let  $C(X)$  denote the space of real continuous functions with the sup norm  $\|\cdot\|$ , and let  $M(X)$  denote the space of Borel probability measures with the topology of weak convergence. Then  $M(X)$  is a compact metric

space. Throughout  $\mu(f)$  denotes the integral of  $f$  against  $\mu$  and all measures are probability measures.

A *Markov semigroup on  $C(X)$*  is a strongly continuous semigroup  $P_t : C(X) \rightarrow C(X)$ ,  $t \geq 0$ , preserving positivity,  $P_t f \geq 0$ , for  $f \geq 0$ , and satisfying  $P_t 1 = 1$ .

The subspace  $\mathcal{D} \subset C(X)$  of functions  $f \in C(X)$  for which the limit

$$(2.1) \quad \left. \frac{d}{dt} \right|_{t=0} P_t f$$

exists in  $C(X)$  is dense. If  $Lf$  is defined to be this limit, then the operator  $L$  is the *generator* of  $P_t$ ,  $t \geq 0$ , on  $C(X)$ .

Given  $V$  in  $C(X)$ , the *Schrodinger semigroup on  $C(X)$*  associated to  $V$  is the unique strongly continuous semigroup  $P_t^V : C(X) \rightarrow C(X)$ ,  $t \geq 0$ , preserving positivity,  $P_t^V f \geq 0$ , for  $f \geq 0$ , with generator  $L + V$ , in the sense the limit

$$(2.2) \quad \left. \frac{d}{dt} \right|_{t=0} P_t^V f$$

exists in  $C(X)$  iff  $f \in \mathcal{D}$ , and equals  $Lf + Vf$ . The Schrodinger semigroup may be constructed as the unique solution of

$$(2.3) \quad P_t^V f = P_t f + \int_0^t P_{t-s} V P_s^V f ds, \quad t \geq 0.$$

for  $f \in C(X)$ . For  $f \geq 0$ , (2.3) implies

$$(2.4) \quad e^{\min V t} P_t f \leq P_t^V f \leq e^{\max V t} P_t f, \quad t \geq 0.$$

Since  $\|P_t^V\| = \max P_t^V 1$ , this implies

$$\min V \leq \lambda_0(V) \leq \max V.$$

For  $\mu$  in  $M(X)$  and  $\lambda_0 = \lambda_0(V)$ , let

$$I^V(\mu) \equiv I(\mu) - \int_X V d\mu + \lambda_0 = - \inf_{u \in \mathcal{D}^+} \int_X \frac{(L + V - \lambda_0)u}{u} d\mu$$

Then  $I^0(\mu) = I(\mu)$  and  $I^V(\mu) = 0$  iff  $\mu$  is an equilibrium measure for  $V$ .

**Lemma 2.1.** *For  $V$  in  $C(X)$ ,  $I^V$  is lower semicontinuous, convex, and nonnegative. In particular,  $I$  is lower semicontinuous, convex, and nonnegative.*

*Proof.* Lower semicontinuity and convexity follow from the fact that  $I^V$  is the supremum of continuous affine functions. The Donsker-Varadhan formula implies  $I^V$  is nonnegative.  $\square$

Let  $L^1(\mu)$  denote the  $\mu$ -integrable Borel functions on  $X$  with norm

$$\|f\|_{L^1(\mu)} = \int_X |f| d\mu = \mu(|f|).$$

For  $V$  in  $C(X)$ , and let  $P_t^V$ ,  $t \geq 0$ , be the Schrodinger semigroup on  $C(X)$ . Let  $\mathcal{D}$  be the domain of the generator  $L$  of  $P_t$ ,  $t \geq 0$ ,  $\mathcal{D}^+$  the positive functions in  $\mathcal{D}$ , and  $C^+(X)$  the positive functions in  $C(X)$ .

By positivity, there is a family  $(t, x) \mapsto p_t^V(x, \cdot)$  of Borel measures on  $X$  such that the Schrodinger semigroup may be written

$$(2.5) \quad P_t^V f(x) = \int_X f(y) p_t^V(x, dy)$$

for  $t \geq 0$ ,  $x \in X$ , and  $f \in C(X)$ . Hence  $P_t^V f(x) \leq +\infty$  is defined for nonnegative Borel  $f$  for all  $t \geq 0$  and  $x \in X$ .

Let  $B(X)$  denote the bounded Borel functions on  $X$  and let  $B^+(X)$  denote the positive Borel functions  $u$  on  $X$  with  $f = \log u \in B(X)$ . We say  $f_n \in B(X)$  converges *boundedly* to  $f \in B(X)$  if  $f_n \rightarrow f$  pointwise everywhere and there is a  $C > 0$  with  $|f_n| \leq C$ ,  $n \geq 1$ .

**Lemma 2.2.** *If  $f_n \rightarrow f$  boundedly then  $u_n = e^{f_n} \rightarrow u = e^f$  boundedly,  $P_t^V u_n \rightarrow P_t^V u$  boundedly, and*

$$\log \left( \frac{e^{-\lambda_0 t} P_t^V u_n}{u_n} \right) \rightarrow \log \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right)$$

*boundedly.*

*Proof.* If  $|f_n| \leq C$ ,  $n \geq 1$ , then  $u_n \rightarrow u$  pointwise. Since  $e^{-C} \leq u_n \leq e^C$ ,  $u_n \rightarrow u$  boundedly. Hence  $P_t^V u_n \rightarrow P_t^V u$  pointwise by the dominated convergence theorem. By (2.4),  $P_t^V u_n \rightarrow P_t^V u$  boundedly. Since

$$\left| \log \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) \right| \leq t(\max V - \min V) + \log \left( \frac{\sup u}{\inf u} \right),$$

the last statement follows.  $\square$

### 3. EQUILIBRIUM MEASURES

**Lemma 3.1.** *For  $u$  in  $B^+(X)$ ,*

$$(3.1) \quad \int_X \log \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu \geq -tI^V(\mu), \quad t \geq 0.$$

*Proof.* By definition of  $I^V(\mu)$ ,

$$(3.2) \quad \int_X \frac{(L + V - \lambda_0)u}{u} d\mu \geq -I^V(\mu), \quad u \in \mathcal{D}^+.$$

For  $t = 0$ , (3.1) is an equality. Moreover for  $t > 0$  and  $u \in \mathcal{D}^+$ , we have  $e^{-\lambda_0 t} P_t^V u \in \mathcal{D}^+$  and

$$\frac{d}{dt} \int_X \log \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu = \int_X \frac{(L + V - \lambda_0)(e^{-\lambda_0 t} P_t^V u)}{e^{-\lambda_0 t} P_t^V u} d\mu \geq -I^V(\mu).$$

This establishes (3.1) for  $u \in \mathcal{D}^+$ . Since  $\mathcal{D}^+$  is dense in  $C^+(X)$ , (3.1) is valid for  $u$  in  $C^+(X)$ .

Now if  $\log u_n \rightarrow \log u$  boundedly and (3.1) holds for  $u_n$ ,  $n \geq 1$ , then by Lemma 2.2, (3.1) holds for  $u$ . Thus the class of Borel functions  $f = \log u$  for which (3.1) holds is closed under bounded convergence. Thus (3.1) holds for all  $u \in B^+(X)$ .  $\square$

The following strengthening of Lemma 3.1 is necessary below.

**Lemma 3.2.** *Let  $u > 0$  Borel satisfy  $\log u \in L^1(\mu)$ . Then for  $t \geq 0$ ,*

$$(3.3) \quad tI^V(\mu) + \int_X \log^+ \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu \geq \int_X \log^- \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu.$$

*Here the integrals may be infinite.*

*Proof.* Without loss of generality, assume  $I^V(\mu) < \infty$ .

Assume in addition  $u \geq \delta > 0$  and let  $u_n = u \wedge n$ ,  $n \geq 1$ . Then  $u_n$  is in  $B^+(X)$ , (3.1) holds with  $u_n$ , and  $u \geq u_n$ , hence

$$\int_X \log \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu \geq \int_X \log \left( \frac{e^{-\lambda_0 t} P_t^V u_n}{u} \right) d\mu \geq - \int_{u > n} \log u d\mu - tI^V(\mu).$$

Letting  $n \rightarrow \infty$  yields (3.1) hence (3.3) for  $\log u$  in  $L^1(\mu)$ , provided  $u \geq \delta > 0$ . Here the left side of (3.3) may be infinite, but the right side is finite.

Now for  $u > 0$  Borel with  $\log u$  in  $L^1(\mu)$ , let  $u_\delta = u \vee \delta$ . Then by what we just derived,

$$tI^V(\mu) + \int_X \log^+ \left( \frac{e^{-\lambda_0 t} P_t^V u_\delta}{u_\delta} \right) d\mu \geq \int_X \log^- \left( \frac{e^{-\lambda_0 t} P_t^V u_\delta}{u_\delta} \right) d\mu$$

so

(3.4)

$$tI^V(\mu) + \int_X \log^+ \left( \frac{e^{-\lambda_0 t} P_t^V u_\delta}{u} \right) d\mu \geq \int_X \log^- \left( \frac{e^{-\lambda_0 t} P_t^V u_\delta}{u} \right) d\mu + \int_{u < \delta} \log u d\mu.$$

We may assume

$$\int_X \log^+ \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu < \infty,$$

otherwise (3.3) is vacuously true. To establish (3.3), we pass to the limit  $\delta \downarrow 0$  in (3.4). Since  $\log u \in L^1(\mu)$  and

$$\log^- \left( \frac{e^{-\lambda_0 t} P_t^V u_\delta}{u} \right), \quad \delta > 0,$$

is an increasing sequence in  $\delta > 0$ , the right side of (3.4) converges to the right side of (3.3) as  $\delta \downarrow 0$ . Since  $u_\delta \leq u + \delta$ ,

$$\log^+ \left( \frac{e^{-\lambda_0 t} P_t^V u_\delta}{u} \right) \leq \log^+ \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) + \log^- u + C, \quad \delta \leq 1,$$

hence the dominated convergence theorem applies. Thus the left side of (3.4) converges to the left side of (3.3) as  $\delta \downarrow 0$ .  $\square$

If  $\psi > 0$  is Borel, for  $u \geq 0$  Borel, define

$$(3.5) \quad P_t^{V,\psi} u \equiv e^{-\lambda_0 t} \frac{P_t^V(\psi u)}{\psi}, \quad t \geq 0.$$

**Lemma 3.3.** Suppose  $\log \psi \in L^1(\mu)$  and let  $u > 0$  Borel satisfy  $\log u \in L^1(\mu)$ . Then for  $t \geq 0$ ,

$$(3.6) \quad tI^V(\mu) + \int_X \log^+ \left( \frac{P_t^{V,\psi} u}{u} \right) d\mu \geq \int_X \log^- \left( \frac{P_t^{V,\psi} u}{u} \right) d\mu.$$

Here the integrals may be infinite.

*Proof.* Since  $\log \psi$  is in  $L^1(\mu)$ ,  $\log(u\psi)$  is in  $L^1(\mu)$  iff  $\log u$  is in  $L^1(\mu)$ . Now apply Lemma 3.2.  $\square$

**Lemma 3.4.** Suppose  $\log \psi \in L^1(\mu)$ . Then  $\mu$  is an equilibrium measure iff for all  $u > 0$  Borel satisfying  $\log u \in L^1(\mu)$ ,

$$\int_X \log^+ \left( \frac{P_t^{V,\psi} u}{u} \right) d\mu \geq \int_X \log^- \left( \frac{P_t^{V,\psi} u}{u} \right) d\mu, \quad t \geq 0.$$

*Proof.* If  $\mu$  is an equilibrium measure,  $I^V(\mu) = 0$  so the result follows from Lemma 3.3. Conversely, assume the inequality holds for all  $u > 0$  satisfying  $\log u \in L^1(\mu)$ . For  $u \in C^+(X)$ , the function  $u/\psi$  satisfies  $\log(u/\psi) \in L^1(\mu)$ . Inserting  $u/\psi$  in the inequality yields

$$\int_X \log^+ \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu \geq \int_X \log^- \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu.$$

For  $u$  in  $C^+(X)$ , these integrals are finite hence

$$\int_X \log \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu \geq 0.$$

Now for  $u \in \mathcal{D}^+$ ,

$$e^{-\lambda_0 t} P_t^V u = u + t(L + V - \lambda_0)u + o(t), \quad t \rightarrow 0,$$

uniformly on  $X$ . Since  $u > 0$ ,

$$\frac{e^{-\lambda_0 t} P_t^V u}{u} = 1 + t \frac{(L + V - \lambda_0)u}{u} + o(t), \quad t \rightarrow 0,$$

uniformly on  $X$ . Thus

$$\log \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) = t \frac{(L + V - \lambda_0)u}{u} + o(t), \quad t \rightarrow 0,$$

uniformly on  $X$ , hence

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_X \log \left( \frac{e^{-\lambda_0 t} P_t^V u}{u} \right) d\mu = \int_X \frac{(L + V - \lambda_0)u}{u} d\mu.$$

This implies

$$\int_X \frac{(L + V - \lambda_0)u}{u} d\mu \geq 0.$$

This implies  $I^V(\mu) \leq 0$ , hence  $I^V(\mu) = 0$ .  $\square$

#### 4. GROUND MEASURES AND GROUND STATES

**Lemma 4.1.** *If  $\pi$  is a ground measure, then  $e^{-\lambda_0 t} P_t^V$ ,  $t \geq 0$ , is a strongly continuous contraction semigroup on  $L^1(\pi)$  and (1.1) holds for  $f$  in  $L^1(\pi)$ .*

*Proof.* (1.1) implies

$$\|e^{-\lambda_0 t} P_t^V f\|_{L^1(\pi)} = \int_X |e^{-\lambda_0 t} P_t^V f| d\pi \leq \int_X e^{-\lambda_0 t} P_t^V |f| d\pi = \int_X |f| d\pi = \|f\|_{L^1(\pi)}$$

for  $f$  in  $C(X)$ . Let  $f$  be in  $L^1(\pi)$  and choose  $f_n \in C(X)$  converging to  $f$  in  $L^1(\pi)$ . By Fatou's lemma,

$$\pi(e^{-\lambda_0 t} P_t^V |f_n - f|) \leq \liminf_m \pi(e^{-\lambda_0 t} P_t^V |f_n - f_m|) = \liminf_m \pi(|f_n - f_m|) = \pi(|f_n - f|).$$

Thus  $P_t^V f^\pm(x) \leq P_t^V |f|(x) < \infty$  for  $\pi$ -a.a.  $x$ ,  $P_t^V f = P_t^V f^+ - P_t^V f^- \in L^1(\pi)$ , and  $P_t^V f_n \rightarrow P_t^V f$  in  $L^1(\pi)$ . Hence  $e^{-\lambda_0 t} P_t^V$ ,  $t \geq 0$ , is a strongly continuous contraction semigroup on  $L^1(\pi)$  satisfying (2.5) for  $f$  nonnegative Borel. The invariance (1.1) for  $f$  in  $L^1(\pi)$  follows.  $\square$

**Lemma 4.2.** *Suppose  $\pi$  and  $\mu$  are measures with  $\mu \ll \pi$ , and let  $\psi = d\mu/d\pi$ . Then  $e^{-\lambda_0 t} P_t^V$ ,  $t \geq 0$ , is a strongly continuous contraction semigroup on  $L^1(\pi)$  iff  $P_t^{V,\psi}$ ,  $t \geq 0$ , is a strongly continuous contraction semigroup on  $L^1(\mu)$ . Moreover, in this case, (1.1) holds for  $f$  in  $L^1(\pi)$  iff*

$$(4.1) \quad \int_X P_t^{V,\psi} f d\mu = \int_X f d\mu, \quad t \geq 0,$$

for  $f$  in  $L^1(\mu)$ .

*Proof.* For  $u \geq 0$  Borel, we have  $\|u\|_{L^1(\mu)} = \|u\psi\|_{L^1(\pi)}$ . Thus  $f \in L^1(\mu)$  iff  $f\psi \in L^1(\pi)$ . If  $f \in L^1(\mu)$ , we also have

$$\|P_t^{V,\psi} f\|_{L^1(\mu)} = \|e^{-\lambda_0 t} P_t^V(f\psi)\|_{L^1(\pi)}$$

and

$$\|P_t^{V,\psi} f - f\|_{L^1(\mu)} = \|e^{-\lambda_0 t} P_t^V(f\psi) - f\psi\|_{L^1(\pi)}.$$

The result is an immediate consequence of these identities.  $\square$

A Markov semigroup on  $L^1(\mu)$  is a strongly continuous contraction semigroup  $Q_t$ ,  $t \geq 0$ , on  $L^1(\mu)$  satisfying  $Q_t f \geq 0$  a.s.  $\mu$  for  $f \geq 0$  a.s.  $\mu$  and  $Q_t 1 = 1$  a.s.  $\mu$ .

**Lemma 4.3.** *Suppose  $\pi$  and  $\mu$  are measures with  $\mu \ll \pi$ , and let  $\psi = d\mu/d\pi$ . Suppose  $\pi$  is a ground measure and  $\psi$  is a ground state relative to  $\mu$ . Then  $P_t^{V,\psi}$ ,  $t \geq 0$ , is a Markov semigroup on  $L^1(\mu)$ .*

*Proof.*  $P_t^{V,\psi} f \geq 0$  a.s.  $\mu$  whenever  $f \geq 0$  a.s.  $\mu$  is clear. If  $\psi$  is a ground state relative to  $\mu$ , then  $P_t^{V,\psi} 1 = 1$  a.s.  $\mu$ . Thus  $P_t^{V,\psi}$ ,  $t \geq 0$ , is a Markov semigroup on  $L^1(\mu)$ .  $\square$

## 5. ENTROPY

For  $\mu, \pi$  in  $M(X)$ , the *entropy* of  $\mu$  relative to  $\pi$  is

$$H(\mu, \pi) \equiv \sup_f \left( \int_X f d\mu - \log \int_X e^f d\pi \right)$$

where the supremum is over  $f$  in  $C(X)$ .

**Lemma 5.1.**  *$H(\mu, \pi) \geq 0$  is finite iff  $\mu \ll \pi$  and  $\psi \log \psi$  is in  $L^1(\pi)$ , where  $\psi = d\mu/d\pi$ , in which case*

$$H(\mu, \pi) = \int_X \psi \log \psi d\pi.$$

Moreover  $H$  is lower-semicontinuous and convex separately in each of  $\mu$  and  $\pi$ .

*Proof.* The lower-semicontinuity and convexity follow from the definition of  $H$  as a supremum of continuous convex functions, in each variable  $\pi$ ,  $\mu$  separately.

Suppose  $H \equiv H(\mu, \pi) < \infty$ ; then

$$(5.1) \quad \int_X f d\mu - \log \int_X e^f d\pi \leq H$$

for  $f$  in  $C(X)$ . The class of Borel functions  $f$  for which (5.1) holds is closed under bounded convergence. Thus (5.1) holds for all  $f \in B(X)$ . Inserting  $f = r1_A$ , where  $\pi(A) = 0$ , we obtain

$$r\mu(A) \leq r\mu(A) - \log(\pi(A^c)) \leq H.$$

Let  $r \rightarrow \infty$  to conclude  $\mu \ll \pi$ . Since  $\psi = d\mu/d\pi \in L^1(\pi)$ , let  $0 \leq f_n \in C(X)$  with  $f_n \rightarrow \psi$  in  $L^1(\pi)$ . By passing to a subsequence, assume  $f_n \rightarrow \psi$  a.s.  $\pi$ . Insert  $f = \log(f_n + \epsilon)$  into (5.1) to yield

$$\int_X \log(f_n + \epsilon) d\mu - \log \int_X (f_n + \epsilon) d\pi \leq H.$$

Let  $n \rightarrow \infty$  followed by  $\epsilon \rightarrow 0$ . Since  $f_n \rightarrow \psi$  in  $L^1(\pi)$ ,

$$\log \int_X (f_n + \epsilon) d\pi \rightarrow \log \int_X \psi d\pi.$$

Thus, by Fatou's lemma (twice),

$$\int_X \psi \log \psi d\pi - \log \int_X \psi d\pi \leq H.$$

Since  $\psi \log^- \psi$  is bounded, this establishes  $\psi \log \psi$  in  $L^1(\pi)$  and  $\int_X \psi \log \psi d\pi \leq H$ .

Conversely, suppose  $\psi = d\mu/d\pi$  exists and  $\psi \log \psi \in L^1(\pi)$ . By Jensen's inequality,

$$\int_X f d\mu \leq \log \int_X e^f d\mu$$

for  $f$  bounded Borel. Replace  $f$  by  $f - \log(\psi \wedge n + \epsilon)$  to get

$$\int_X f d\mu - \log \int_X \left( \frac{e^f \psi}{\psi \wedge n + \epsilon} \right) d\pi \leq \int_X \psi \log(\psi \wedge n + \epsilon) d\pi.$$

Let  $\epsilon \rightarrow 0$  followed by  $n \rightarrow \infty$  obtaining

$$\int_X f d\mu - \log \int_X e^f d\pi \leq \int_X \psi \log \psi d\pi.$$

Now maximize over  $f$  in  $C(X)$  to conclude  $H(\mu, \pi) \leq \int_X \psi \log \psi d\pi$ .  $\square$

## 6. PROOFS OF THE THEOREMS

*Proof of Theorem 1.* Assume  $\pi$  is a ground measure and  $\psi$  is a ground state relative to  $\mu$ . By Lemma 4.3,  $P_t^{V,\psi}$ ,  $t \geq 0$ , is a Markov semigroup on  $L^1(\mu)$ . Suppose  $\log u \in L^1(\mu)$ . Then  $P_t^{V,\psi}(\log u)$  is in  $L^1(\mu)$ , hence there is a set  $N$  with  $\mu(N) = 0$  and  $P_t^{V,\psi}(|\log u|)(x) < \infty$  and  $P_t^{V,\psi}1(x) = 1$  for  $x \notin N$ . Jensen's inequality applied to the integral  $f \mapsto (P_t^{V,\psi}f)(x)$  implies

$$\log \left( \frac{P_t^{V,\psi}u}{u} \right) (x) \geq P_t^{V,\psi}(\log u)(x) - (\log u)(x), \quad x \notin N.$$

Thus the negative part of

$$(6.1) \quad \log \left( \frac{P_t^{V,\psi}u}{u} \right)$$

is in  $L^1(\mu)$  and

$$\int_X \log \left( \frac{P_t^{V,\psi}u}{u} \right) d\mu \geq \int_X \left( P_t^{V,\psi}(\log u) - \log u \right) d\mu = 0.$$

By Lemma 3.4, this implies  $\mu$  is an equilibrium measure, establishing the first claim.

Assume  $\pi$  is a ground measure and  $\mu$  is an equilibrium measure. Then  $e^{-\lambda_0 t} P_t^V$ ,  $t \geq 0$ , is a strongly continuous contraction semigroup on  $L^1(\pi)$ , hence  $P_t^{V,\psi}$ ,  $t \geq 0$ ,



is a strongly continuous contraction semigroup on  $L^1(\mu)$ . Since  $1 \in L^1(\mu)$ ,  $P_t^{V,\psi} 1 \in L^1(\mu)$  hence  $\log^+(P_t^{V,\psi} 1) \in L^1(\mu)$ . By Lemma 3.4,  $\log^-(P_t^{V,\psi} 1) \in L^1(\mu)$  hence  $\log(P_t^{V,\psi} 1) \in L^1(\mu)$ . By Jensen's inequality, (4.1), and Lemma 3.4,

$$0 = \log(\mu(1)) = \log\left(\int_X P_t^{V,\psi} 1 \, d\mu\right) \geq \int_X \log(P_t^{V,\psi} 1) \, d\mu \geq 0.$$

Since  $\log$  is strictly concave, this can only happen if  $P_t^{V,\psi} 1$  is  $\mu$  a.s. constant. By (4.1), the constant is 1. Since  $\psi > 0$  a.s.  $\mu$  is immediate, this establishes the second claim.

Assume  $\mu$  is an equilibrium measure and  $\psi$  is a ground state relative to  $\mu$ . Then  $P_t^{V,\psi} 1 = 1$  a.s.  $\mu$ , hence for  $u \in B^+(X)$ ,

$$\frac{\inf u}{\sup u} \leq \frac{P_t^{V,\psi} u}{u} \leq \frac{\sup u}{\inf u}, \quad a.s. \mu,$$

hence (6.1) is in  $L^\infty(\mu)$ . Thus by (3.6), for  $f \in B(X)$ ,

$$\int_X \log\left(\frac{P_t^{V,\psi} e^{\epsilon f}}{e^{\epsilon f}}\right) d\mu \geq 0, \quad \epsilon > 0.$$

Since  $P_t^{V,\psi} 1 = 1$  a.s.  $\mu$  and  $e^{\epsilon f} = 1 + \epsilon f + o(\epsilon)$ ,

$$\frac{1}{\epsilon} \log\left(\frac{P_t^{V,\psi} e^{\epsilon f}}{e^{\epsilon f}}\right) \rightarrow P_t^{V,\psi} f - f, \quad \epsilon \rightarrow 0,$$

uniformly a.s.  $\mu$  on  $X$ . Thus

$$\int_X (P_t^{V,\psi} f - f) \, d\mu \geq 0$$

for  $f \in B(X)$ . Applying this to  $\pm f$  yields

$$(6.2) \quad \int_X e^{-\lambda_0 t} P_t^V(\psi f) \, d\pi = \int_X \psi f \, d\pi$$

for  $f \in B(X)$ . Now let  $\psi_\delta = (\psi \vee \delta) \wedge (1/\delta)$ . Then  $\psi/\psi_\delta \rightarrow 1$  pointwise. For  $f$  in  $C(X)$ ,  $f/\psi_\delta \in B(X)$ ; inserting this into (6.2) yields

$$(6.3) \quad \int_X e^{-\lambda_0 t} P_t^V\left(\frac{\psi f}{\psi_\delta}\right) d\pi = \int_X \frac{\psi f}{\psi_\delta} d\pi.$$

Since

$$\frac{|f|\psi}{\psi_\delta} \leq |f| + |f|\psi, \quad \delta \leq 1,$$

sending  $\delta \rightarrow 0$  in (6.3) yields (1.1). Hence  $\pi$  is a ground measure, establishing the third claim.  $\square$

*Proof of Theorem 2.* Let

$$M \equiv \sup_{t \geq 0} e^{-\lambda_0 t} \|P_t^V\|.$$

By (3.1),

$$\int_X \log\left(\frac{e^{-\lambda_0 t} P_t^V u}{u}\right) d\mu \geq -tI^V(\mu), \quad u \in C^+(X).$$

Thus for  $f \in C(X)$ ,

$$\int_X f d\mu - \int_X \log(e^{-\lambda_0 t} P_t^V e^f) d\mu \leq tI^V(\mu), \quad f \in C(X).$$

By Jensen's inequality,

$$\int_X f d\mu - \log \int_X (e^{-\lambda_0 t} P_t^V e^f) d\mu \leq tI^V(\mu), \quad f \in C(X).$$

Defining

$$Z_t \equiv e^{-\lambda_0 t} \mu(P_t^V 1)$$

and

$$\pi_t(f) \equiv \frac{e^{-\lambda_0 t} \mu(P_t^V f)}{Z_t}$$

yields

$$\int_X f d\mu - \log \int_X e^f d\pi_t \leq tI^V(\mu) + \log Z_t, \quad f \in C(X).$$

Taking the supremum over all  $f$  yields

$$H(\mu, \pi_t) \leq tI^V(\mu) + \log Z_t.$$

Note  $Z_t \leq M$ ,  $t \geq 0$ , hence

$$H(\mu, \pi_t) \leq tI^V(\mu) + \log M, \quad t \geq 0.$$

Now set

$$\bar{\pi}_T(f) \equiv \frac{\int_0^T Z_t \pi_t(f) dt}{\int_0^T Z_t dt} = \frac{\int_0^T e^{-\lambda_0 t} \mu(P_t^V f) dt}{\int_0^T Z_t dt}, \quad T > 0.$$

Then  $\pi_t$  is in  $M(X)$  for  $t > 0$  and  $\bar{\pi}_T$  is in  $M(X)$  for  $T > 0$ .

Now assume  $\mu$  is an equilibrium measure; then  $I^V(\mu) = 0$ . By convexity of  $H$ ,

$$H(\mu, \bar{\pi}_T) \leq \log M, \quad T > 0.$$

By compactness of  $M(X)$ , select a sequence  $T_n \rightarrow \infty$  with  $\pi_n = \bar{\pi}_{T_n}$  converging to some  $\pi$ . By lower-semicontinuity of  $H$ , we have  $H(\mu, \pi) \leq \log M$ . Thus  $\mu \ll \pi$  with  $\psi = d\mu/d\pi$  satisfying  $\psi \log \psi \in L^1(\pi)$ . By Lemma 3.1,

$$\log Z_t = \log \mu(e^{-\lambda_0 t} P_t^V 1) \geq \mu(\log(e^{-\lambda_0 t} P_t^V 1)) \geq 0,$$

hence  $Z_t \geq 1$ ,  $t \geq 0$ . Since

$$e^{-\lambda_0 t} \mu(P_t^V (L + V - \lambda_0)f) = \frac{d}{dt} e^{-\lambda_0 t} \mu(P_t^V f), \quad f \in \mathcal{D},$$

we have

$$|\bar{\pi}_T((L + V - \lambda_0)f)| = \frac{|e^{-\lambda_0 T} \mu(P_T^V f) - \mu(f)|}{\int_0^T Z_t dt} \leq \frac{2M\|f\|}{T} \rightarrow 0, \quad T \rightarrow \infty.$$

Thus

$$\pi((L + V - \lambda_0)f) = 0, \quad f \in \mathcal{D},$$

which implies

$$e^{-\lambda_0 t} \pi(P_t^V f) = \pi(f), \quad t \geq 0,$$

for  $f \in \mathcal{D}$ , hence, by density, for  $f$  in  $C(X)$ . Thus  $\pi$  is a ground measure.

Clearly  $\log \psi \in L^1(\mu)$  and  $\psi > 0$  a.s.  $\mu$ . Since  $\mu$  is an equilibrium measure and  $\pi$  is a ground measure, Theorem 1 implies  $\psi$  is a ground state relative to  $\mu$ .  $\square$

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DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122  
*E-mail address:* [hijab@temple.edu](mailto:hijab@temple.edu)